JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **31**, No. 4, November 2018
http://dx.doi.org/10.14403/jcms.2018.31.1.487

CONSISTENCY AND GENERAL TRUNCATED MOMENT PROBLEMS

Seonguk Yoo*

Abstract. The Truncated Moment Problem (TMP) entails finding a positive Borel measure to represent all moments in a finite sequence as an integral; once the sequence admits one or more such measures, it is known that at least one of the measures must be finitely atomic with positive densities (equivalently, a linear combination of Dirac point masses with positive coefficients). On the contrary, there are more general moment problems for which we aim to find a "signed" measure to represent a sequence; that is, the measure may have some negative densities. This type of problem is referred to as the General Truncated Moment Problem (GTMP). The Jordan Decomposition Theorem states that any (signed) measure can be written as a difference of two positive measures, and hence, in the view of this theorem, we are able to apply results for TMP to study GTMP. In this note we observe differences between TMP and GTMP; for example, we cannot have an analogous to the Flat Extension Theorem for GTMP. We then present concrete solutions to lower-degree problems.

1. Introduction

The moment problem is a special class of the inverse problem which naturally occurs in many areas of science and mathematics. While the classical theory of moments has been studied for over a century, the systematic study of truncated moment problems began only two decades ago.

Received October 23, 2018; Accepted November 07, 2018.

2010 Mathematics Subject Classification: Primary 44A60, 47A57; Secondary 11P70, 15-04, 47A20, 32A60.

Key words and phrases: truncated moment problem, signed measure, algebraic variety, consistency.

*Supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2016R1A6A3A11932349).

We now state the definition of the (bivariate) truncated moment problem: Consider a doubly indexed finite sequence of real numbers (of order $2n$), $\beta \equiv \beta^{(2n)} = \{\beta_{00},\ \beta_{10},\ \beta_{01}, \cdots, \beta_{2n,0},\ \beta_{2n-1,1}, \cdots, \beta_{1,2n-1}, \beta_{0,2n}\}$ with $\beta_{00} \neq 0$. We call the terms in the sequence *moments* inspired by T. J. Stieltjes. The truncated real moment problem entails to find a positive Borel measure μ supported in the real plane $\mathbb{R}^2$ such that

$$\beta_{ij} = \int x^i y^j \, d\mu \ \ (i, j \in \mathbb{Z}_+,\ 0 \leq i + j \leq 2n). \tag{1.1}$$

The measure μ is referred to as a *representing measure* for β. In a similar way, the *full* moment problem for an infinite sequence β^{∞} is defined. J. Stochel showed that β^{∞} has a representing measure on a closed set K if and only if $\beta^{(2n)}$ has a representing measure supported in K for each n [24]. Thus, it is necessary to find a solution of TMP for all orders to solve the full moment problem. On the other hand, the general truncated moment problem calls for a (signed) measure satisfying (1.1); to avoid any confusion, a representing measure for TMP is referred to as a "positive" measure through this article and one for GTMP is said to be a measure. As a univariate moment problem, R. Boas showed any infinite sequence of real numbers admits a signed measure supported in $[0, \infty)$ [4]. Even though he did not discuss, it seems possible to find an algorithm for a formula of a measure through the construction in the proof of Boas' result. However, there is still no well-formulated algorithm to cover this issue for both infinite or finite moment sequences; that is, even under the presence of measures, we do not know how to write a closed form of a measure.

For better understanding of GTMP, we need to review TMP and the following are basic notions about TMP.

1.1. Definitions and Notations

We now define the *(real) moment matrix* $M(n) \equiv M(n)(\beta)$ of β as below: $M(n)$ is a square matrix of size $\binom{n+2}{2}$. The rows and columns of $M(n)$ are labeled with monomials in the following lexicographical order: $\mathit{1}, X, Y, X^2, XY, Y^2, \cdots, X^n, \cdots, Y^n$. The entry in the row X^iY^j and the column X^kY^l is the moment $\beta_{i+k,j+l}$, for example, $M(2)$ is written as

$$\begin{array}{cc} & \begin{array}{cccccc} \mathit{1} & X & Y & X^2 & XY & Y^2 \end{array} \\ \begin{array}{c} \mathit{1} \\ X \\ Y \\ X^2 \\ XY \\ Y^2 \end{array} & \left(\begin{array}{c|cc|ccc} \beta_{00} & \beta_{10} & \beta_{01} & \beta_{20} & \beta_{11} & \beta_{02} \\ \hline \beta_{10} & \beta_{20} & \beta_{11} & \beta_{30} & \beta_{21} & \beta_{12} \\ \beta_{01} & \beta_{11} & \beta_{02} & \beta_{21} & \beta_{12} & \beta_{03} \\ \hline \beta_{20} & \beta_{30} & \beta_{21} & \beta_{40} & \beta_{31} & \beta_{22} \\ \beta_{11} & \beta_{21} & \beta_{12} & \beta_{31} & \beta_{22} & \beta_{13} \\ \beta_{02} & \beta_{12} & \beta_{03} & \beta_{22} & \beta_{13} & \beta_{04} \end{array}\right) \end{array}$$

Notice that $M(n)$ is a symmetric, bolck Hankel matrix. We will see algebraic properties of $M(n)$ make great contributions for the existence of a positive representing measure for β.

We will write column dependence relations in $M(n)$ using these labels and will see that the relations behave like polynomials, and produce important necessary conditions for the existence of a positive measure.

In detail, we define an assignment from $\mathcal{P}_n$ (the set of 2-variable polynomials whose degree is at most n) to $\mathcal{C}_{M(n)}$ (the column space of $M(n)$); given a polynomial $p(x,y) \equiv \sum_{ij} a_{ij}x^iy^j$, we let $p(X,Y) := \sum_{ij} a_{ij}X^iY^j$ (so that $p(X,Y) \in \mathcal{C}_{M(n)}$). This map will deliver useful information for TMP; for example, the column relations will be used to identify the location of the support of a positive representing measure (see Subsection 1.2).

Through the Jordan Decomposition Theorem, we may adopt the results for TMP to study GTMP and for the perspective, basic necessary conditions for TMP are to be introduced briefly:

1.2. Necessary Conditions for TMP

If β has a positive representing measure μ, then we can compute that for a polynomial $p(x,y) = \sum_{i,\,j=0}^{n} a_{ij}x^iy^j$

$$0 \le \int p(x,y)^2\, d\mu = \sum_{i,j,k,l} a_{ij}a_{kl} \int x^{i+k}y^{j+l}\, d\mu = \sum_{i,j,k,l} a_{ij}a_{kl}\beta_{i+k,\,j+l},$$

which is equivalent that $M(n)$ is positive (semidefinite), denoted as $M(n) \ge 0$. Even though it was the most basic, important necessary condition for TMP, we obviously know that the positivity is no longer required for GTMP.

Let $\mathcal{Z}(p)$ denote the zero set of a polynomial p and define the *algebraic variety* $\mathcal{V}$ of β or $M(n)$ by

$$\mathcal{V} \equiv \mathcal{V}(\beta) \equiv \mathcal{V}(M(n)) := \bigcap_{p \in \mathcal{P}_n,\, p(X,Y)=\mathbf{0}} \mathcal{Z}(p). \tag{1.2}$$

For the presence of a positive measure, it must be true from Proposition 3.1 in [7] that supp $\mu \subseteq \mathcal{V}(\beta)$ and this inclusion determines where atoms of a positive measure lie. While we cannot enjoy this inequality for GTMP (that is, some atoms of a measure can be outside of the algebraic variety; see the example in Section 2.2), it is still possible to find a measure supported in the algebraic variety of GTMP (see Subsection 2.1) and we will see concrete conditions for the existence of a measure whose support is in the algebraic variety. For computing the algebraic variety, we note that if $\widehat{p}$ denotes the column vector of coefficients of $p(x, y)$, then we know $p(X, Y) = M(n)\widehat{p}$; in other words, $p(X, Y) = \mathbf{0}$ if and only if $\widehat{p} \in \ker\ M(n)$. Thus, we run the Gauss elimination with $M(n)$ and diagnose the column relations; after regarding them as polynomials, we need to find the common zero set of the polynomials.

Another necessary condition called the *variety condition* is that rank $M(n) \leq$ card $\mathcal{V}$; we will prove this is valid for GTMP as well (see Proposition 2.2). We need more necessary conditions to study a higher order moment problem:

Definition 1.1. Let $M(n)$ be a moment matrix with the algebraic variety $\mathcal{V} \equiv \mathcal{V}(M(n))$. Also, let $\Lambda \equiv \Lambda_\beta$ be the *Riesz functional* that maps a polynomial $p(x, y) := \sum_{i,j=0}^{n} a_{ij} x^i y^j \in \mathcal{P}_{2n}$ to the linear combination of moments $\sum_{i,j=0}^{n} a_{ij}\beta_{ij}$. Then we say that:

(i) β or $M(n)$ is **weakly consistent (on $\mathcal{V}$)** if $p \in \mathcal{P}_n$ and $p|_{\mathcal{V}} \equiv 0$, then $\Lambda(p) = 0$;
(ii) β or $M(n)$ is **consistent (on $\mathcal{V}$)** if $p \in \mathcal{P}_{2n}$ and $p|_{\mathcal{V}} \equiv 0$, then $\Lambda(p) = 0$.

The main results in [10] says that the consistency on $\mathcal{V}$ is sufficient for the *extremal* case (rank $M(n) =$ card $\mathcal{V}$) and also we know that the consistency cannot be replaced by the weaker necessary condition that $M(n)$ is *recursively generated*; that is,

$$\text{if } p(X,Y) = \mathbf{0}, \text{then } (p\,q)(X,Y) = \mathbf{0}, \text{ for all } q \text{ with } \deg(p\,q) \leq n. \tag{RG}$$

More importantly for this article, the consistency is known as a necessary and sufficient condition to have a measure; however, checking the consistency cannot be done in a simple manner. In Section 5 we will establish the consistency of the quartic ($n = 2$) GTMP with the aid of the Division Algorithm.

1.3. Flat Extension

Probably, the most efficient and concrete solution to TMP would be the Flat Extension Theorem [7, Theorem 5.13]. It says that a positive $M(n)$ admits a *flat* (rank-preserving, positive) extension $M(n+1)$ if and only if β has a positive rank $M(n)$-atomic measure. Thus, we build a positive extension allowing several new moments (parameters) and try to keep its rank the same. Observe that an moment matrix extension $M(n+1)$ can be written as $M(n+1) = \begin{pmatrix} M(n) & B \\ B^* & C \end{pmatrix}$, for some matrices B and C.

Furthermore, we can find atoms and densities of a positive representing measure explicitly via the results related to the Flat Extension Theorem [7]. For a solution of GTMP we are naturally attracted to consider that flatness of $M(n)$ may imply the consistency of $M(n)$. We have to know, however, flatness cannot guarantee the existence of a measure supported in the algebraic variety (see Section 2.2).

Note also that it looks easy to find a flat extension in principle but keeping the moment matrix structure in $M(n+1)$ brings many difficulties; although we may use a computer algebra like *Mathematica* or *Matlab*, the construction of $M(n+1)$ is not feasible for most TMP of $n \geq 3$ due to memory overflow.

2. The Consistency and a Measure

We now discuss topics related to the consistency of a moment matrix and investigate properties of a measure of a consistent moment sequence. The next Lemma shows that the consistency is strong enough to yield a (finitely atomic) measure, which may have some negative densities.

Lemma 2.1. [10, Lemma 2.3] *Let $L : \mathcal{P}_{2n} \to \mathbb{R}$ be a linear functional and let $V \subseteq \mathbb{R}^2$. The following statements are equivalent:*

(i) There exist $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$ and there exist $w_1, \ldots, w_m \in V$ such that

$$L(p) = \sum_{i=1}^{m} \alpha_i p(w_i) \tag{2.1}$$

for all $p \in \mathcal{P}_{2n}$.

(ii) If $p \in \mathcal{P}_{2n}$ and $p|_V \equiv 0$, then $L(p) = 0$.

If L is the Riesz functional of the moment sequence β and if V is the algebraic variety of $M(n)$, then Lemma 2.1(ii) is just the consistency condition. While it seems like Lemma 2.1 gives a concrete solution to GTMP, we should indicate that checking the consistency is a highly nontrivial process. We have to see that the Riesz functional of all the polynomials vanishing on the algebraic variety is assigned to zero. This test requires certain techniques to represent all the polynomials that vanish on the algebraic variety.

The next result shows an import necessary condition for TMP, the variety condition, is also valid for GTMP:

PROPOSITION 2.2. *If β is consistent on the nonempty $\mathcal{V}(\beta)$, then* rank $M(n)(\beta) \leq$ card $\mathcal{V}(\beta)$.

Proof. Let $\mathcal{V} = \{(x_i, y_i)\}_{i=1}^{v}$ for some $v > 0$ and denote the generalized Vandermonde matrix as $\mathcal{E}_{\mathcal{V}}$; that is,

$$\mathcal{E}_{\mathcal{V}} = \begin{pmatrix} 1 & x_1 & y_1 & x_1^2 & x_1y_1 & y_1^2 & \cdots & x_1^n & \cdots & y_1^n \\ 1 & x_2 & y_2 & x_2^2 & x_2y_2 & y_2^2 & \cdots & x_2^n & \cdots & y_2^n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & x_v & y_v & x_v^2 & x_vy_v & y_v^2 & \cdots & x_v^n & \cdots & y_v^n \end{pmatrix} \tag{2.2}$$

In the same fashion as we did for $M(n)$, we label the columns of $\mathcal{E}_{\mathcal{V}}$ with monomials in the lexicographical order: $\mathit{1}, X, Y, X^2, XY, Y^2, \ldots, X^n, \ldots, Y^n$. If rank $M(n) = r$, the column relations in $M(n)$ can be written as $p_1(X,Y) = \mathbf{0}, \ldots, p_\eta(X,Y) = \mathbf{0}$, where $\eta = (n+1)(n+2)/2 - r$. Notice that all of the column relations $p_1(X,Y) = \mathbf{0}, \ldots, p_\eta(X,Y) = \mathbf{0}$ must hold in $\mathcal{E}_{\mathcal{V}}$ as well.

Suppose $r > v$. Then $\mathcal{E}_{\mathcal{V}}$ must have additional column relation other than $p_i(X,Y) = \mathbf{0}$'s. Thus, there is a new polynomial $q(x,y) \in \mathcal{P}_n$ that vanishes on $\mathcal{V}$. Observe that the leading term of $q(X,Y)$ is not in the basis of the column space of $M(n)$. Since β is consistent on $\mathcal{V}(\beta)$, we know that $\Lambda(x^i y^j q) = 0$ for $i, j \in \mathbb{Z}_+$ and $0 \leq i + j \leq n$, which brings another column relation $q(X,Y) = \mathbf{0}$ in $\mathcal{C}_{M(n)}$, and hence the rank of $M(n)$ becomes $r - 1$. This is a contradiction. $\square$

2.1. Jordan Decomposition Theorem

The Jordan decomposition theorem states that every measure μ has a unique decomposition into a difference $\mu = \mu^+ - \mu^-$ of two positive measures μ^+ and μ^-, at least one of which is finite. After rearranging the terms in (2.1) by the sign of densities, we can write a measure of a consistent $M(n)$ as

$$\mu = \sum_{k=1}^{\ell} \rho_k \delta_{w_k} - \sum_{k=\ell+1}^{s} \rho_k \delta_{w_k}, \tag{2.3}$$

where $\rho_k > 0$ for all $k = 1, \ldots, \ell$; we denote the first summand in (2.3) as μ^+ and the second as μ^-. Due to this fact, a bound of the cardinality of the support of a measure is established:

Proposition 2.3. *A minimal measure for β which is consistent on $\mathcal{V}(\beta)$ is at most $(2n+1)(2n+2)$-atomic.*

Proof. If $M(n)$ is consistent with a measure $\mu = \mu^+ - \mu^-$ of two positive (finite) measures μ^+ and μ^-, we may write $M(n) = M[\mu^+] - M[\mu^-]$, where each term is a moment matrix generated by the corresponding positive measure of the same size as $M(n)$. A result [2, Theorem 2] for TMP found by C. Bayer and J. Teichmann showed that the cardinality of the support of a positive measure is at most $\dim \mathcal{P}_{2n}$ in the presence of a positive measure for a moment matrix whose order is n. Since $M[\mu^+]$ and $M[\mu^-]$ have a positive measure, it follows that a minimal measure for each moment matrix is at most $\dim \mathcal{P}_{2n}$ atomic. Therefore, we conclude that card supp $\mu \le 2(\dim \mathcal{P}_{2n}) = (2n+1)(2n+2)$. □

2.2. Differences between TMP and GTMP

Recall that in the presence of a positive representing measure μ for a positive $M(n)(\beta)$, Proposition 3.1 in [7] states that

$$\hat{p} \in \ker M(n)(\beta) \iff p(X,Y) = \mathbf{0} \iff \text{supp } \mu \subseteq \mathcal{Z}(p).$$

This result tells us that the atoms of a positive measure μ must be contained in $\mathcal{V}(M(n))$, but the following example shows such an argument is no longer valid for GTMP; consider

$$M(1) \equiv M(1)\left(\beta^{(2)}\right) = \begin{pmatrix} -1 & -16 & -4 \\ -16 & -94 & -10 \\ -4 & -10 & 2 \end{pmatrix}. \tag{2.4}$$

$M(1)$ has a single column relation $Y = -(4/3)\mathit{1} + (1/3)X$ and we can easily check by the upcoming Lemma 3.5 that $\beta^{(2)}$ is consistent on $\mathcal{V}(M(1))$, and hence we can find a measure supported in $\mathcal{Z}(p) \subseteq \mathcal{V}$; will readily see this claim in Section 4. However, one of representing measures for $M(1)$ is $\nu = \delta_{(-2,1)} + \delta_{(-2,-2)} - \delta_{(1,1)} - \delta_{(10,1)}$ but supp $\nu \not\subseteq \mathcal{Z}(y + 4/3 - (1/3)x)$. In other words, a measure may have atoms outside of the algebraic variety. This example motivates us to find some conditions when a moment sequence has a measure supported in the algebraic variety. Main results in Section 5 will show such conditions.

We recall that a positive (semidefinite) $M(n)$ is a flat extension of a positive $M(n-1)$ if and only if $M(n)$ has a unique positive rank $M(n)$-atomic representing measure. However, the coming example shows flatness of $M(n)$ does not guarantee the existence of a measure supported in $\mathcal{V}$ for GTMP; consider

$$M(2) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & -1 & 0 & 2 \end{pmatrix} \tag{2.5}$$

$$\xrightarrow{\text{row reduction}} \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We see rank $M(1) =$ rank $M(2) = 3$ and $M(2)$ has 3 column relations: $X^2 = 1$, $XY = -Y$, and $Y^2 = -\mathit{1} + X$. Since the 3 polynomials of column relations intersect only at $(1, 0)$, it follows that $M(2)$ does not satisfy the inequality in Proposition 2.2; hence, $M(2)$ is not consistent on $\mathcal{V}(M(2))$ (equivalently, it has no measure whose support is contained in $\mathcal{V}(M(2))$) even if it is flat. This example suggests that building a rank-preserving extension of $M(n)$ would not be adaptable to solve GTMP while it is the most efficient tool for TMP. We may also use a direct approach to see $M(2)$ does not admit a measure supported in $\mathcal{V}$: It is not difficult to check that all the polynomials vanishing on the variety $\{(1, 0)\}$ could be generated by $x - 1$ and y only. For the consistency of $M(2)$, we must have $\Lambda(x^i y^j (x - 1)) = 0$ and $\Lambda(x^i y^j \cdot y) = 0$ for $i, j \in \mathbb{Z}_+$, $0 \le i + j \le 3$ but this is not true; explicitly, $\Lambda(x - 1) = \beta_{10} - \beta_{00} = -1 \neq 0$.

2.3. Degree-One Transformations

We introduce an affine transformation (called the *degree-one transformation*) through which we can classify TMP's as a group of equivalent curves. For example, any moment matrix with a conic column relation can be translated into a moment matrix with one of the column relations

$X^2 + Y^2 = \mathit{1}$, $X^2 - Y^2 = \mathit{1}$, $Y = X^2$, $XY = \mathit{1}$, or $XY = \mathbf{0}$ as in [8]. Hence, it allows us to consider much smaller groups of TMP; even we can make a given TMP much simpler with an appropriate transformation (see [8]). We can show that the degree-one transformation is valid for GTMP as well. The complex version of the transformation is described in [8, Proposition 1.7] and it is also known, in the same paper, that there is an equivalence between the complex truncated moment problem and the real truncated moment problem; thus, we may use the real version of the transformation for the study of GTMP. Here is the description of the degree-one transformation of the real version; for $a, b, c, d, e, f \in \mathbb{R}$, $ae - bd \neq 0$, let $\varphi_1(x, y) := ax + by + c$, $\varphi_2(x, y) := dx + ey + f$, and $\varphi(x, y) := (\varphi_1(x, y), \varphi_2(x, y))$. Given a real moment sequence $\beta \equiv \beta^{(2n)}$, define a new moment sequence $\tilde{\beta}$ by $\tilde{\beta}_{ij} := \Lambda_\beta(\varphi_1^i \varphi_2^j)$ $(i, j \in \mathbb{Z}_+$, $0 \leq i + j \leq 2n)$, where Λ_β denotes the Riesz functional associated with β. We can easily verify that $\Lambda_{\tilde{\beta}}(p) = \Lambda_\beta (p \circ \varphi)$ for every $p \in \mathcal{P}_{2n}$ and the invariance between moment problems is described as follows:

Proposition 2.4. [8, cf. Proposition 1.7] (Invariance under degree-one transformations.) *Let $M(n)$ and $\tilde{M}(n)$ be the moment matrices associated with β and $\tilde{\beta}$, and let $J\hat{p} := \widehat{p \circ \varphi}$ $(p \in \mathcal{P}_{2n})$. Then the following are true:*

(i) $\tilde{M}(n) = J^T M(n) J$;
(ii) *J is invertible;*
(iii) *rank $\tilde{M}(n) =$ rank $M(n)$;*
(iv) *The formula $\mu = \tilde{\mu} \circ \varphi$ establishes a one-to-one correspondence between the sets of measures for β and $\tilde{\beta}$, which preserves measure class and cardinality of the support; moreover, $\varphi(\text{supp } \mu) = \text{supp } \tilde{\mu}$;*
(v) *For $p \in \mathcal{P}_n$, $p(\tilde{X}, \tilde{Y}) = J^T ((p \circ \varphi)(X, Y))$.*

The final topic of this section is the Division Algorithm from the real algebraic geometry. The algorithm will make main contributions to the results in Section 4 and 5. We will be able to construct a structure theorem for polynomials that vanish on the algebraic variety.

Theorem 2.5 (Division Algorithm). [5] *Fix a monomial order $>$ on $\mathbb{Z}_+^n$, and let $F = (f_1, \cdots, f_s)$ be an ordered s-tuple of polynomials in $\mathbb{R}[x_1, \cdots, x_n]$. Then every $f \in \mathbb{R}[x_1, \cdots, x_n]$ can be written as*

$$f = a_1 f_1 + \cdots + a_s f_s + r,$$

where $a_i, r \in \mathbb{R}[x_1, \cdots, x_n]$, and either $r = 0$ or r is a linear combination of monomials with coefficients in $\mathbb{R}$, none of which is divisible by any *leading terms of* $f_1, \cdots, f_s$. *We call* r *a* remainder *of* f *on division by* F. *Furthermore, if* $a_i f_i \neq 0$, *then we have*

$$\text{multideg}(f) \geq \text{multideg}(a_i f_i).$$

3. Vandermonde Matrix Method to Check the Consistency

In this section, we will collect earlier methods developed for checking the consistency since these results provide an insight into a new approach using the Division Algorithm. We adopt the notations in [14] and remark that these arguments cover only bivariate moment problems even though the result works for multidimensional moment problems:

- Columns of $M(n)$ are labeled with the monomials $X^iY^j \in \mathcal{P}_n$ ($i, j \in \mathbb{Z}_+$, $0 \leq i + j \leq n$) in degree-lexicographical order;
- $V := \{(x_1, y_1), \ldots, (x_s, y_s)\}$ is a finite subset in $\mathbb{R}^2$;
- $W_m[V]$ is a matrix with s rows and with columns labeled with X^iY^j. (Note that the entry of $W_m[V]$ in the row k ($1 \leq k \leq s$) and the column X^iY^j is $x_k^i y_k^j$, and hence $W_m[V]$ is a Vandermonde matrix of points in $\mathbb{R}^2$);
- $U_m[V] := W_m[V]^T$;
- $\tau(m) := \dim \mathcal{P}_m = \binom{m+2}{m}$;

Given $M(n)(\beta)$, let $\tau \equiv \tau(2n)$, $r = \text{rank } M(n)$, $v = \text{card } \mathcal{V}(M(n))$, and set $L_\beta := (\beta_{00}, \beta_{10}, \beta_{01}, \ldots, \beta_{2n,0}, \ldots, \beta_{0,2n})^T \in \mathbb{R}^\tau$. Let $\mathcal{B}$ denote a basis of $\mathcal{C}_{M(n)}$, the column space of $M(n)$. For the case when $V \subseteq \mathcal{V}(M(n))$, let $W_{\mathcal{B}}[V]$ denote the compression of $W_{2n}[V]$ to columns in $\mathcal{B}$ and let $U_{\mathcal{B}}[V] \equiv W_{\mathcal{B}}[V]^T$. After understanding the proof of Lemma 2.1 carefully, we can get the following results:

Theorem 3.1. [14, Theorem 3.2] *If a positive* $M(n)$ *has a flat extension, then* $U_{\mathcal{B}}[V]$ *is invertible for a subset* V *of* $\mathcal{V}(M(n))$.

Proposition 3.2. [14, Proposition 3.6 and 3.8] *Let* $v < \infty$. *Then*

(i) β *is consistent if and only if* $L_\beta \in \text{Ran } U_{2n}[\mathcal{V}(M(n))]$.
(ii) β *is weakly consistent if and only if* $\text{Ran}\, M(n) \subseteq \text{Ran}\, U_n[\mathcal{V}(M(n))]$; *equivalently, there exists a matrix* Z *such that* $M(n) = U_n[\mathcal{V}(M(n))]Z$.

For the extremal moment problems ($r = v$), we have a useful way to check weakly consistency and it is to be used for a main result later:

Lemma 3.3. [10, Lemma 4.1] *The following are equivalent for an extremal* β*:*

(i) $p \in \mathcal{P}_n$ *and* $p|_{\mathcal{V}} \equiv 0 \implies p(X,Y) = \mathbf{0}$ *in* $\mathcal{C}_{M(n)}$*;*
(ii) For any basis $\mathcal{B}$ *of* $\mathcal{C}_{M(n)}$*,* $U_{\mathcal{B}}$ *is invertible;*
(iii) There exists a basis $\mathcal{B}$ *of* $\mathcal{C}_{M(n)}$ *such that* $U_{\mathcal{B}}$ *is invertible.*

An important necessary condition (RG) is somewhat rigid in a sense; once there is a lower-order column relation, many of higher-order relations have to be determined. For some cases, all the column relations may be originated from a single polynomial p and the algebraic variety is just $\mathcal{Z}(p)$; the case ends up with an infinite variety. Moreover, if p is irreducible, the consistency of $M(n)$ is immediate as follows:

Definition 3.4. For $p \in \mathcal{P}_n$, $M(n)$ is *p-pure* if the only dependence relation in $\mathcal{C}_{M(n)}$ are those of the form $(pq)(X,Y) = \mathbf{0}$ for some $q \in \mathcal{P}_{n-\deg p}$.

Lemma 3.5. [15, Lemma 3.1] $\beta \equiv \beta^{(2n)}$ *is consistent if* $M(n)$ *is p-pure, where* $\deg p(x,y) = n$*, p is irreducible in* $\mathbb{R}[x,y]$*, and* $\mathcal{Z}(p)$ *is infinite.*

In summary, if the algebraic variety is finite or if all the column relations are determined by an irreducible polynomial, then the preceding results would be applicable.

4. General Quadratic Moment Problem

In order to discuss a way of writing a formula of a measure, we consider the rank-one decomposition method. We try to decompose a consistent moment matrix as a linear combination of rank-one moment matrices. Indeed, if $M(n)$ is consistent with an r-atomic measure, then we may write

$$M(n) = \sum_{i=1}^{r} \rho_i \mathbf{v}_i \mathbf{v}_i^T,$$

where $\rho_i \in \mathbb{R}$, $\mathbf{v}_i := (1, x_i, y_i, \ldots, x_i^n, x_i^{n-1} y_i, \ldots, \; x_i y_i^{n-1}, y_i^n)^T$, and $(x_i, y_i) \in \mathbb{R}^2$ for $i = 1, \ldots, r$.

We now classify $M(1)$ in terms of its rank and present a solution:

Theorem 4.1. *Suppose* $\beta_{00} \neq 0$ *and* rank $M(1) = r$*. Then:*
(i) $r = 1$ *if and only if* $M(1)$ *admits a unique 1-atomic measure;*
(ii) $r = 2$ *if and only if* $M(1)$ *admits a 2-atomic measure;*
(iii) If $r = 3$*, then* $M(1)$ *admits a 3-atomic measure.*

Proof. (i) The backward implication is trivial and to prove the forward; suppose rank $M(1) = 1$. Without loss of generality, we may assume the columns X and Y are linearly dependent on the first column *1*. Now we can write

$$M(1) = \begin{pmatrix} \beta_{00} & \beta_{10} & \beta_{01} \\ \beta_{10} & \beta_{20} & \beta_{11} \\ \beta_{01} & \beta_{11} & \beta_{02} \end{pmatrix} = \beta_{00} \begin{pmatrix} 1 & a & b \\ a & c & d \\ b & d & e \end{pmatrix},$$

where a, b, c, d, and e are obviously defined. Since the column X in $M(1)$ is dependent on *1*, the column relation must be $X = a\mathit{1}$, which implies that $c = a^2$ and $d = ab$; similarly, the other column relation must be $Y = b\mathit{1}$ and this relation determines that $d = b^2$. Thus, $M(1) = \beta_{00}(1, a, b)^T(1, a, b)$ and the sequence has the unique measure $\beta_{00}\,\delta_{(a,b)}$.

(ii) The sufficiency is clear and we try to prove the necessity. Since rank $M(1) = 2$, we can find a linear column relation $p(X, Y) = \mathbf{0}$ in $\mathcal{C}_{M(1)}$. We observe $M(1)$ is p-pure and from Lemma 3.5 we know that $M(1)$ admits a measure. Applying the invariance under the degree-one transformation, it is possible to convert the linear column relation $p(X, Y) = \mathbf{0}$ to $Y = \mathbf{0}$; that is, it is enough to consider the case

$$M(1) = \begin{pmatrix} \beta_{00} & \beta_{10} & 0 \\ \beta_{10} & \beta_{20} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We use the rank-one decomposition method to see $M(1)$ has a 2-atomic measure. The algebraic variety of $M(1)$ is the line $y = 0$ and we select a point from the line, say $(x_1, 0)$. Consider a column vector $\mathbf{v}_1 = (1, x_1, 0)^T$ and we build a rank-one moment matrix $P_1 := \mathbf{v}_1\mathbf{v}_1^T$. Now we define $\widetilde{M(1)} := M(1) - \rho_1 P_1$ for some $\rho_1 \in \mathbb{R}$. In order to have a minimal (rank $M(2)$-atomic) representing measure, we force rank $\widetilde{M(1)}$ to be 1; a calculation shows $\rho_1 = (\beta_{10}^2 - \beta_{00}\beta_{20})/(-\beta_{20} + 2\beta_{10}x_1 - \beta_{00}x_1^2)$ is such a choice. Since the rank of $\widetilde{M(1)}$ is now 1, we can apply the proof (i) and get that the unique representing measure of $\widetilde{M(1)}$ is $\rho_2\delta_{(x_2,0)}$, where $\rho_2 = (\beta_{10}^2 - 2\beta_{00}\beta_{10}x_1 + \beta_{00}^2x_1^2)/(\beta_{20} - 2\beta_{10}x_1 + \beta_{00}x_1^2)$ and $x_2 = (\beta_{20} - \beta_{10}x_1)/(\beta_{10} - \beta_{00}x_1)$. Thus $M(1)$ has a 2-atomic measure $\rho_1\delta_{(x_1,0)} + \rho_2\delta_{(x_2,0)}$, where x_1 must be different from a solution of the equations:

$$\beta_{20} - 2\beta_{10}x_1 + \beta_{00}x_1^2 = 0 \text{ and } \beta_{10} - \beta_{00}x_1 = 0.$$

(iii) A nonsingular moment matrix has no column relation, which means the algebraic variety is the entire plane. The only polynomial vanishes throughout the plane is the zero polynomial; the consistency is established immediately from this fact.

To find a 3-atomic measure, we get through 2 steps of the rank reduction. We first take an arbitrary point (p,q) (indeed, $(0,0)$ would be efficient) and represent $M(1)$ as

$$M(1) = \widetilde{M} + uM[\delta_{(p,q)}],$$

where $M[\delta_{(p,q)}]$ is a moment matrix generated by $\delta_{(p,q)}$. Find then a value of u such that rank $\widetilde{M(1)} = 2$; once you identity a column relation in $\widetilde{M(1)}$, the rest work is entirely similar to the computation given in the proof of (ii).

These complete the proof of Theorem 4.1. $\square$

Example 4.2. *We illustrates how to find a measure of the sequence in (2.4):*

$$M(1) = \begin{pmatrix} -1 & -16 & -4 \\ -16 & -94 & -10 \\ -4 & -10 & 2 \end{pmatrix}$$

Observe that rank $M(1) = 2$ *and the unique column relation is* $Y = (-4/3)\mathit{1} + (1/3)X$*; by Theorem 4.1 (ii),* $M(1)$ *has a 2-atomic measure. We would like to see there are measures supported in the algebraic variety of* $M(1)$*. Selecting a point* $(a, (a-4)/3)$ *for some real* a*, we may write*

$$M(1) = \widetilde{M(1)} + u \begin{pmatrix} 1 & a & \frac{a-4}{3} \\ a & a^2 & \frac{a(a-4)}{3} \\ \frac{a-4}{3} & \frac{a(a-4)}{3} & \frac{(a-4)^2}{9} \end{pmatrix} \tag{4.1}$$

for some u*. In order to have a minimal (*rank $M(2)$*-atomic) representing measure, we must have* rank $\widetilde{M(1)} = 1$*; a calculation shows* rank $\widetilde{M(1)} = 1$ *if and only if* $u = 162/(a^2 - 32a + 94)$*. If we take*

$u = 162/(a^2 - 32a + 94)$, *then*

$$M(1) = \frac{-(a-16)^2}{a^2-32a+94}\begin{pmatrix} 1 & \frac{2(8a-47)}{a-16} & \frac{2(2a-5)}{a-16} \\ \frac{2(8a-47)}{a-16} & \frac{4(8a-47)^2}{(a-16)^2} & \frac{4(2a-5)(8a-47)}{(a-16)^2} \\ \frac{2(2a-5)}{a-16} & \frac{4(2a-5)(8a-47)}{(a-16)^2} & \frac{4(2a-5)^2}{(a-16)^2} \end{pmatrix} + \frac{162}{a^2-32a+94}\begin{pmatrix} 1 & a & \frac{a-4}{3} \\ a & a^2 & \frac{a(a-4)}{3} \\ \frac{a-4}{3} & \frac{a(a-4)}{3} & \frac{(a-4)^2}{9} \end{pmatrix}.$$

We just now got the minimal measure $\mu = \frac{-(a-16)^2}{a^2-32a+94}\delta_{\left(\frac{2(8a-47)}{a-16}, \frac{2(2a-5)}{a-16}\right)} + \frac{162}{a^2-32a+94}\delta_{\left(a, \frac{a-4}{3}\right)}$.

5. General Quartic Moment Problem

We again classify $M(2)$ with its rank and present conditions about $M(2)$ when it admits a measure supported in $\mathcal{V}(M(2))$. In the sequel, we assume $\beta_{00} \neq 0$.

5.1. The Case When rank $M(2) = 1$

We can verify that the (RG)-condition is sufficient for the existence of the 1-atomic measure. For, if $X = A\mathit{1}$ and $Y = B\mathit{1}$ are the two linear column relations in $M(2)$, then the conic relations must be $X^2 = A^2\mathit{1}$, $XY = AB\mathit{1}$, and $Y^2 = B^2\mathit{1}$. We can imitate the proof of the case when rank $M(1) = 1$ and confirm the result.

5.2. The Case When rank $M(2) = 2$

Since (RG) is a necessary condition for the existence of a measure supported in $\mathcal{V}(M(2))$, the moment matrix $M(2)$ cannot help but have a linear column relation. The only possible two cases are that the basis of $\mathcal{C}_{M(2)}$ is $\{\mathit{1}, X\}$ or $\{\mathit{1}, Y\}$. By virtue of the degree-one transformation $X = \widetilde{Y}$ and $Y = \widetilde{X}$, we know that the moment problems with these two bases are equivalent; it is sufficient to consider the first case, and so we assume now the basis of $\mathcal{C}_{M(2)}$ is $\{\mathit{1}, X\}$. Since the column X^2 is linearly dependent, we write $X^2 = k_1\mathit{1} + k_2X$ for some $k_1, k_2 \in \mathbb{R}$. In order to satisfy the variety condition, the zero set of the corresponding polynomial to the column relation must be a product of two vertical lines. In other words, the quadratic equation $x^2 = k_1 + k_2x$ must have two different roots. We may write the column relation in X^2 as in the following result for computational ease:

THEOREM 5.1. *Let $\{1, X\}$ be the basis of $\mathcal{C}_{M(2)}$. If the first two column relations are written as $Y = A1 + BX$ and $X^2 = (C+D)X - (CD)1$ with $(C+D)^2 - 4CD > 0$ and $C \neq D$, then $M(2)$ has a measure supported in $\mathcal{V}(M(2))$ if and only if $M(2)$ is (RG); that is, the other two column relations in $\mathcal{C}_{M(2)}$ are $XY = AX + BX^2$ and $Y^2 = AY + BXY$.*

Proof. ($\Longrightarrow$) It is trivial since the consistency implies (RG).
($\Longleftarrow$) We first observe that the algebraic variety is

$$\mathcal{V} := \{(C, A + BC), (D, A + BD)\}.$$

To establish the consistency of $M(2)$, assume $p \in \mathcal{P}_4$ and $p|_{\mathcal{V}} \equiv 0$. Also let $q_1(x,y) := y - (A + Bx)$ and $q_2(x,y) := x^2 - (C+D)x + CD$. Since $M(2)$ is (RG) and the column relations in $M(2)$ allow us to get

$$\Lambda(x^i y^j q_1) = 0 \qquad \text{for } 0 \leq i + j \leq 3; \tag{5.1}$$

$$\Lambda(x^k y^\ell q_2) = 0 \qquad \text{for } 0 \leq k + \ell \leq 2. \tag{5.2}$$

Using the Division Algorithm, we may write

$$p(x,y) = p_1(x,y)q_1(x,y) + p_2(x,y)q_2(x,y) + r(x,y)$$

where $p_1(x,y) \in \mathcal{P}_3$, $p_2(x,y) \in \mathcal{P}_2$, and $r(x,y) := \alpha_0 + \alpha_1 x$ for some $\alpha_0, \alpha_1 \in \mathbb{R}$. Since $r(C, A+BC) = 0$ and $r(D, A+BD) = 0$, it follows that $\alpha_0 = \alpha_1 = 0$ under the condition $C \neq D$. This shows r is the zero polynomial and the consistency of $M(2)$ is obtained by (5.1) and (5.2). $\square$

5.3. The Case When rank $M(2) = 3$

The (RG)-condition about $M(2)$ is essential so that we may have only 3 possibilities; the basis of the column space would be one of the following:

$$\{1, X, Y\}, \qquad \{1, Y, Y^2\}, \qquad \{1, X, X^2\}.$$

The first case is covered by the proceeding result:

THEOREM 5.2. *Let $\{1, X, Y\}$ be the basis of $\mathcal{C}_{M(2)}$. Then $M(2)$ has a measure supported in $\mathcal{V}(M(2))$ if and only if $M(2)$ satisfies the variety condition.*

Proof. The forward implication is obvious; we begin with denoting the conic column relation in $M(2)$ as:

$$\begin{aligned} q_1(X,Y) &\equiv X^2 + \ell_1(X,Y) = \mathbf{0}; \\ q_2(X,Y) &\equiv XY + \ell_2(X,Y) = \mathbf{0}; \\ q_3(X,Y) &\equiv Y^2 + \ell_3(X,Y) = \mathbf{0}, \end{aligned}$$

where $\ell_i(x,y)$'s are some linear polynomials. Suppose $p \in \mathcal{P}_4$ and $p|_{\mathcal{V}} \equiv 0$, where $\mathcal{V}(M(2))$ is the algebraic variety of $M(n)$. Since $M(2)$ satisfies the variety condition, we know $3 \leq \text{card } \mathcal{V}$. Now the Division Algorithm enables us to get

$$p(x,y) = p_1(x,y)q_1(x,y) + p_2(x,y)q_2(x,y) + p_3(x,y)q_3(x,y) + r(x,y),$$

where $p_1, p_2, p_3 \in \mathcal{P}_2$, and $r(x,y) := \alpha_0 + \alpha_1 x + \alpha_2 y$ for some real numbers α_0, α_1, and α_2. Since $p|_{\mathcal{V}} \equiv 0$ if and only if $r|_{\mathcal{V}} \equiv 0$, it follows that the line $r(x,y) = 0$ must intersect with conic $q_i(x,y) = 0$'s at least 3 different points; however, it cannot happen by the Bézout Theorem. Thus, we conclude that r is the zero polynomial and the consistency is naturally gained by the 3 column relations in $M(2)$. □

We then observe the later two cases are equivalent through the degree-one transformation of interchanging x and y. We just need to cover the second case:

THEOREM 5.3. *Let $\{\mathit{1}, Y, Y^2\}$ be the basis of $\mathcal{C}_{M(2)}$. Then $M(2)$ has a measure supported in $\mathcal{V}(M(2))$ if and only if $M(2)$ is recursively generated.*

Proof. It suffices to prove the backward implication. Because $M(2)$ is recursively generated, we know the column relations in $M(2)$ should be:

$$\begin{aligned} q_1(X,Y) &\equiv X - A\mathit{1} = \mathbf{0}; \\ q_2(X,Y) &\equiv X^2 - A^2\mathit{1} = \mathbf{0}; \\ q_3(X,Y) &\equiv XY - AY = \mathbf{0}, \end{aligned}$$

where A is a real number. The algebraic variety $\mathcal{V}(M(2))$ is the set of all the points on the line $x = A$; if $p \in \mathcal{P}_4$ and $p|_{\mathcal{V}} \equiv 0$, then application of the Division Algorithm gives us

$$p(x,y) = p_1(x,y)q_1(x,y) + p_2(x,y)q_2(x,y) + p_3(x,y)q_3(x,y) + r(x,y),$$

where $p_1 \in \mathcal{P}_3$, $p_2, p_3 \in \mathcal{P}_2$, and $r(x,y) := \alpha_0 + \alpha_1 y + \alpha_2 y^2 + \alpha_3 y^3 + \alpha_4 y^4$ for some real numbers $\alpha_0, \alpha_1, \ldots, \alpha_4$. The graph of the remainder r also contains all the points on $x = A$; however, it is impossible if r has a nonzero coefficient. Thus, we see r must be the zero polynomial. The consistency just follows from the 3 column relations in $M(2)$. $\square$

5.4. The Case When rank $M(2) = 4$

Each case in the category has exactly 2 conic column relations. We know that the possible bases of $\mathcal{C}_{M(2)}$ are:

$$\mathcal{B}_1 := \left\{\mathit{1}, X, Y, X^2\right\}, \quad \mathcal{B}_2 := \left\{\mathit{1}, X, Y, XY\right\}, \quad \mathcal{B}_3 := \left\{\mathit{1}, X, Y, Y^2\right\}.$$

Unless the two conics of the column relations have a common factor, it follows from Bézout's Theorem that card $\mathcal{V} \leq 4$. Thus, it is necessary to get exactly 4 points in the variety to satisfy the variety condition; indeed, the problem becomes extremal ($r = v$). The possibility of an infinite variety is in the cases of $\mathcal{B}_1$ and $\mathcal{B}_3$ only. $M(2)$ with the basis $\mathcal{B}_2$ cannot admit a variety with infinitely many points (we will verify this in the proof of Theorem 5.6). Notice also that the cases of $\mathcal{B}_1$ and $\mathcal{B}_3$ are equivalent under the degree-one transformation of interchanging x and y; hence, it is enough to cover the cases with $\mathcal{B}_1$ and $\mathcal{B}_2$.

Theorem 5.4. *If $\mathcal{B}_1$ is the basis of $\mathcal{C}_{M(2)}$ and* card $\mathcal{V} = \infty$, *then $M(2)$ has a measure supported in $\mathcal{V}(M(2))$.*

Proof. In oder to have an infinite variety, the polynomials of the conic column relations must be factored as a pair of lines and share a line as intersection. If $p_1(X, Y) = \mathbf{0}$ and $p_2(X, Y) = \mathbf{0}$ are the two column relations, then we may write the associated polynomials as

$$p_1(x, y) = (y + ax + b)(y + cx + d) \quad \text{and} \quad p_2(x, y) = (y + ax + b)(x + e)$$

for some $a, b, c, d, e \in \mathbb{R}$. Note that since the leading term of one quadratic polynomial has to be xy, the second factor of p_2 does not have any y variable. We then use the invariance of the moment problem under a degree-one transformation and deal with an simpler, equivalent problem; if we take $\tilde{x} = x$ and $\tilde{y} = y + ax + b$, then we can show that the equivalent moment matrix $\widetilde{M(2)}$ has the column relations, $Y^2 + (-a + c)XY + (-b + d)Y = \mathbf{0}$ and $XY + eY = \mathbf{0}$ (via this transformation we may consider the common factor as y instead of $y + ax + b$). This transition enables us to assume that the algebraic variety is the set of all the points in $y = 0$ and $(-e, (c - a)e - d + b)$ which stays outside of $y = 0$; that is, $(c - a)e - d + b \neq 0$.

Now let $\tilde{q}_1(x, y) := y^2 + (-a + c)xy + (-b + d)y$ and $\tilde{q}_2(x, y) := xy + ey$. For checking the consistency, suppose $p \in \mathcal{P}_4$ and $p|_{\tilde{\mathcal{V}}} \equiv 0$, where $\tilde{\mathcal{V}}$ is the algebraic variety of $\widetilde{M(n)}$. The Division Algorithm allows us to have the representation of p:

$$p(x,y) = p_1(x,y)\tilde{q}_1(x,y) + p_2(x,y)\tilde{q}_2(x,y) + r(x,y),$$

where $p_1, p_2 \in \mathcal{P}_2$, and $r(x,y) := \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4 + \alpha_5 y$ for $\alpha_0, \ldots, \alpha_5 \in \mathbb{R}$. Since $r(0,0) = 0$, we determine $\alpha_0 = 0$. The polynomial $r(x,y)$ is divisible by y, and hence $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$; the final request is that $r(-e,(c-a)e-d+b)) = \alpha_5[(c-a)e-d+b] = 0$. Thus, α_5 is zero and r is indeed the zero polynomial. The consistency is checked by the two column relations promptly. □

For the case with the basis $\mathcal{B}_1$ and card $\mathcal{V} = 4$, we follow a similar argument in Theorem 5.4. Let us write the column relations as $p_1(X,Y) = XY - a_0 \mathit{1} - a_1 X - a_2 Y - a_3 X^2$ and $p_2(X,Y) = Y^2 - b_0 \mathit{1} - b_1 X - b_2 Y - b_3 X^2$ for $a_0, \ldots, a_3, b_0, \ldots, b_3 \in \mathbb{R}$. Assume $p \in \mathcal{P}_4$ and $p|_{\mathcal{V}} \equiv 0$. Through the Division Algorithm, we get

$$p(x,y) = p_1(x,y)q_1(x,y) + p_2(x,y)q_2(x,y) + r(x,y),$$

where $p_1, p_2 \in \mathcal{P}_2$, and $r(x,y) := \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 x^2 + \alpha_4 x^3 + \alpha_5 x^4$ for $\alpha_0, \ldots, \alpha_5 \in \mathbb{R}$.

Suppose the variety $\mathcal{V}(M(2))$ is explicitly known by solving the system $p_1(x,y) = 0$ and $p_2(x,y) = 0$, say $\mathcal{V} = \{(x_1,y_1),(x_2,y_2),(x_3,y_3),(x_4,y_4)\}$. We label the columns of the Vandermonde matrix $\mathcal{E}_{\mathcal{V}}$ of $\mathcal{V}(M(2))$ as we did for $M(2)$. Because $M(2)$ is necessarily to be weakly consistent, the restriction of $\mathcal{E}_{\mathcal{V}}$ to a basis of $\mathcal{C}_{M(2)}$ is invertible by Lemma 3.3; that is, the first four columns in $\mathcal{E}_{\mathcal{V}}$ are linearly independent. We now get, for some k_i, ℓ_i and $i = 1, \ldots, 4$:

(5.3)

$$\mathcal{E}'_{\mathcal{V}} := \begin{pmatrix} 1 & x_1 & y_1 & x_1^2 & x_1^3 & x_1^4 \\ 1 & x_2 & y_2 & x_2^2 & x_2^3 & x_2^4 \\ 1 & x_3 & y_3 & x_3^2 & x_3^3 & x_3^4 \\ 1 & x_4 & y_4 & x_4^2 & x_4^3 & x_4^4 \end{pmatrix} \xrightarrow{\text{row reduction}} \begin{pmatrix} 1 & 0 & 0 & 0 & k_1 & \ell_1 \\ 0 & 1 & 0 & 0 & k_2 & \ell_2 \\ 0 & 0 & 1 & 0 & k_3 & \ell_3 \\ 0 & 0 & 0 & 1 & k_4 & \ell_4 \end{pmatrix}.$$

For r to vanish on $\mathcal{V}(M(2))$, the equation $\mathcal{E}'_{\mathcal{V}}(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)^T = \mathbf{0}$ has at least one solution; equivalently, r can be rewritten as

$$\begin{aligned} (5.4)\quad r(x,y) = &-(k_1\alpha_4 + \ell_1\alpha_5) - (k_2\alpha_4 + \ell_2\alpha_5)x - (k_3\alpha_4 + \ell_3\alpha_5)y \\ &-(k_4\alpha_4 + \ell_4\alpha_5)x^2 + \alpha_4 x^3 + \alpha_5 x^4, \end{aligned}$$

where α_4 and α_5 are real parameters. Beyond conics arising from the column relations, we establish the consistency by testing $\Lambda(r) = 0$. Indeed, we just now proved the next result:

Theorem 5.5. *Suppose* $\mathcal{B}_1$ *is the basis of* $\mathcal{C}_{M(2)}$ *and* card $\mathcal{V} = 4$. *Then* $M(2)$ *has a measure supported in* $\mathcal{V}(M(2))$ *if and only if the restriction of the Vandermonde matrix* $\mathcal{E}_{\mathcal{V}}$ *to a basis of* $\mathcal{C}_{M(2)}$ *is invertible and* $\Lambda(r) = 0$, *where* r *is the polynomial in (5.4).*

The final case of rank $M(2) = 4$ is to be solved:

Theorem 5.6. *Let* $\mathcal{B}_2$ *be the basis of* $\mathcal{C}_{M(2)}$. *Suppose* $M(2)$ *satisfies the variety condition; that is,* card $\mathcal{V}(M(2)) = 4$. *Then* $M(2)$ *has a measure supported in* $\mathcal{V}(M(2))$ *if and only if the restriction of the Vandermonde matrix* $\mathcal{E}_{\mathcal{V}}$ *to a basis of* $\mathcal{C}_{M(2)}$ *is invertible.*

Proof. Since the consistency implies weak the consistency which is equivalent to invertibility of the restricted Vandermonde matrix, the backward implication is clear. For the other implication, observe that the two conic column relations in $M(2)$ should be:

$$\begin{aligned} q_1(X,Y) &\equiv X^2 + \ell_1(X,Y) = \mathbf{0}; \\ q_2(X,Y) &\equiv Y^2 + AXY + \ell_2(X,Y) = \mathbf{0}, \end{aligned}$$

where ℓ_1 and ℓ_2 are some linear polynomials, and $A \in \mathbb{R}$. We now claim that the algebraic variety must be finite. For, suppose not. Then the two polynomials share a line as intersection. Since the leading term of q_2 is y^2, its two factors must have the term y. If q_1 is reducible and one of its factor has the term y, then it is impossible to avoid the term xy in q_1. This is a contradiction. By the Bézout's Theorem, we know that card $\mathcal{V} \leq 4$.

Now set $p \in \mathcal{P}_4$ and $p|_{\mathcal{V}} \equiv 0$; use the Division Algorithm to represent p as:

$$p(x,y) = p_1(x,y)q_1(x,y) + p_2(x,y)q_2(x,y) + r(x,y),$$

where $p_1, p_2 \in \mathcal{P}_2$, and $r(x,y) := \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 xy$ for $\alpha_0, \ldots, \alpha_3 \in \mathbb{R}$. Set the variety $\mathcal{V} = \{(s_i,t_i)\}_{i=1}^4$; the equations $r(s_i,t_i) = 0$ for $i = 1, \ldots, 4$ are written as a matrix multiplication:

$$\begin{pmatrix} 1 & x_1 & y_1 & x_1y_1 \\ 1 & x_2 & y_2 & x_2y_2 \\ 1 & x_3 & y_3 & x_3y_3 \\ 1 & x_4 & y_4 & x_4y_4 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{5.5}$$

Since the above Vandermonde matrix is invertible, the zero vector is the unique solution of the matrix equation; in other words, $r(x,y) \equiv 0$. For checking the consistency of $M(2)$, we just need to take care of multiples of q_1 and q_2; the two column relations guarantee all the required tests. $\square$

5.5. The Case When rank $M(2) = 5$ or 6

Note that if $M(2)$ has a linear column relation and rank $M(2) = 5$, then $M(2)$ is not recursively generated; thus, it has no measure supported in $\mathcal{V}(M(2))$. $M(2)$ necessarily has a conic column relation to admit a measure and the next result shows it is enough.

THEOREM 5.7. *If a unique column relation in $M(2)$ is a conic column relation, then $M(2)$ has a measure supported in $\mathcal{V}(M(2))$.*

Proof. If $M(2) \equiv M(2)(\beta)$ has a single conic column relation and the associated polynomial is irreducible, then the consistency of the moment sequence is simply obtained by Lemma 3.5. On the other hand, if $M(2)$ has a reducible column relation $p(X, Y) = \mathbf{0}$, then the conic $p(x, y) = 0$ has two linear factors. Suppose $r(x, y) \in \mathcal{P}_4$ with $r|_{\mathcal{V}} \equiv 0$. Since both factors of $p(x, y)$ are necessary in the representation of the algebraic variety $\mathcal{V} = \mathcal{Z}(p)$, it follows from [23, Proposition 3.3] that there exists $q(x, y) \in \mathbb{R}[x, y]$ such that $r = pq$, and hence $\deg q = \deg r - \deg p \leq n$. Now we observe $\Lambda_\beta(r) = \Lambda_\beta(pq) = \langle M(2)\hat{p}, \hat{q}\rangle = \langle \mathbf{0}, \hat{q}\rangle = 0$, which proves that $M(2)$ is consistent. $\square$

For the invertible $M(2)$, as explained in the proof of Theorem 4.1 (iii), the moment sequence is naturally consistent.

6. Conclusions and Remarks

In this article, we have presented complete solutions of GTMP when $n = 1$ or 2. If $n \geq 3$, then the problems become much more difficult to handle like TMP; moreover, a solution seems to require a numerical condition as in Theorem 5.5 or solutions of TMP for $n = 3$ [12, 13]. It is not obvious how and why such a numerical condition contributes for the existence of a measure; we leave it for future study. Moreover, we may consider problems of higher order. The author believes that the method used in solutions of extremal sextic TMP in [12, 13] could be applicable for the purpose of general cases.

Before we conclude the article, several unanswered questions will be collected in this section. One important part of moment problems is to determine uniqueness of a measure for a moment sequence; if a moment sequence has a unique measure, then it is said to be *determinate.* We thus naturally ask:

QUESTION 6.1. When does $M(n)$ have a unique measure supported in $\mathcal{V}(M(n))$?

An answer to this question for TMP was flatness of a positive $M(n)$. However, most examples in this note (except the cases of rank $M(1) =$ rank $M(2) = 1$) have infinitely many measures. We may conjecture that $M(n)$ is not determinate of rank $M(n) > 1$.

The next is related to the number and the location of atoms of a measure μ; we have a bound of card supp μ suggested in Section 2, but it seems to be somewhat blunt. All the examples considered so far have the bound, rank $M(n)$, which is dominated by the cardinality of the algebraic variety $\mathcal{V}$. Being relevant to TMP, we may consider:

QUESTION 6.2. For a minimal measure μ for $M(n)$, can we have the inequality, card supp $\mu \leq$ rank $M(n)$?

As we saw an example in Section 2.2, the number of atoms of a measure can be strictly greater than rank $M(n)$ and the atoms lie in somewhere other than the algebraic variety. However, we might ask that:

QUESTION 6.3. If μ is a minimal measure for $M(n)$, then does $\mathcal{V}(M(n))$ contain supp μ?

Acknowledgments

The author is indebted to Professors Raúl Curto and Sang Hoon Lee for several discussions related to the topics in this note. The author is also deeply grateful to the referee for many suggestions that led to significant improvements in the presentation. Most of the examples, and some of the proofs in this paper, were obtained with the computer algebra *Mathematica* [27].

References

[1] N. I. Akhiezer, *The classical moment problem and some related questions in analysis*, Hafner Publishing Co. New York, 1965.

[2] C. Bayer and J. Teichmann, *The proof of Tchakaloff's Theorem*, Proc. Amer. Math. Soc. **134** (2006), 3035-3040.

[3] G. Blekherman, *Nonnegative polynomials and sums of squares*, J. Amer. Math. Soc. **25** (2012), 617-635.

[4] R. P. Boas, *The Stieltjes moment problem for functions of bounded variation*, Bull. Amer. Math. Soc. **45** (1939), no. 6, 399-404.

[5] D. Cox, J. Little, and D. O'Shea, *Ideals, varieties and algorithms: an introduction to computational algebraic geometry and commutative algebra*, Second Edition, Springer-Verlag, New York, 1992.

[6] R. Curto and L. Fialkow, *Recursiveness, positivity, and truncated moment problems*, Houston J. Math. **17** (1991), no. 4, 603-635.

[7] R. Curto and L. Fialkow, *Solution of the truncated complex moment problem with flat data*, Memoirs Amer. Math. Soc. no. 568, Amer. Math. Soc. Providence, 1996.

[8] R. Curto and L. Fialkow, *Solution of the singular quartic moment problem*, J. Operator Theory, **48** (2002), 315-354.

[9] R. Curto and L. Fialkow, *An analogue of the Riesz-Haviland theorem for the truncated moment problem*, J. Funct. Anal. **255** (2008), no. 10, 2709-2731.

[10] R. Curto, L. Fialkow, and H. M. Möller, *The extremal truncated moment problem*, Integral Equations Operator Theory, **60** (2008), no. 2, 177-200.

[11] R. Curto, S. H. Lee, and J. Yoon, *A new approach to the 2-variable subnormal completion problem*, J. Math. Anal. Appl. **370** (2010), no. 1, 270-283.

[12] R. Curto and S. Yoo, *Cubic column relations in truncated moment problems*, J. Funct. Anal. **266** (2014), no. 3, 1611-1626.

[13] R. Curto and S. Yoo, *Concrete solution to the nonsingular quartic binary moment problem*, Proc. Amer. Math. Soc. **144** (2016), no. 1, 249-258.

[14] L. Fialkow, *Truncated multivariable moment problems with finite variety*, J. Operator Theory, **60** (2008), no. 2, 343-377.

[15] L. Fialkow, *Solution of the truncated moment problem with variety $y = x^3$*, Trans. Amer. Math. Soc. **363** (2011), 3133-3165

[16] L. Fialkow and J. Nie, *Positivity of Riesz functionals and solutions of quadratic and quartic moment problems*, J. Funct. Anal. **258** (2010), no. 1, 328-356.

[17] M. Putinar and F.-H. Vasilescu, *Solving moment problems by dimensional extension*, Annals of Math. **149** (1999), 1087-1107.

[18] C. Scheiderer, *Sums of squares on real algebraic curves*, Math. Z. **245** (2003), no. 4, 725-760.

[19] K. Schmüdgen, *The K-moment problem for compact semi-algebraic sets*, Math. Ann. **289** (1991), no. 2, 203-206.

[20] K. Schmüdgen, *On the moment problem of closed semi-algebraic sets*, J. Reine Angew. Math. **558** (2003), 225-234.

[21] J. A. Shohat and J. D. Tamarkin, *The Problem of Moments*, American Mathematical Society Mathematical surveys, **vol. I.** American Mathematical Society, New York, 1943.

[22] J. L. Smul'jan, *An operator Hellinger integral (Russian)*, Mat. Sb. **49(91)** (1959), no. 4, 381-430.

[23] J. Stochel, *Moment functions on real algebraic sets*, Ark. Mat. **30** (1992), 133-148.

[24] J. Stochel, *Solving the truncated moment problem solves the full moment problem*, Glasg. Math. J. **43** (2001), no. 3, 335-341.

[25] J. Stochel and F. Szafraniec, *Algebraic operators and moments on algebraic sets*, Portugal. Math. **51** (1994), no. 1, 25-45.

[26] S. Yoo, *Extremal sextic truncated moment problems*, (Ph.D.) Thesis, The University of Iowa, 2010, 149 pages.

[27] Wolfram Research, Inc. *Mathematica*, **Version 11.0**, Champaign, IL, 2009.

*

Department of Mathematics Education and RINS
Gyeongsang National University
Jinju 52828, Republic of Korea
E-mail: seyoo@gnu.ac.kr